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Scaling and the quantum Hall effect

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Abstract. The description of the quantum Hall effect in terms of a two-parameter scaling theory is considered by studying the results of numerical simulations. It is argued that the apparently contradictory results can be explained in terms of a modification of the flow diagram predicted by Pruisken and co-workers. This modification is shown to be consistent with the generalised flow diagram for the fractional effect.

1. Introduction

The scaling description of the localisation of electrons in disordered systems (Abrahams *et al* 1979), which has been so successful in describing the transport properties of quasi two-dimensional systems in the absence of a magnetic field, remains controversial when it is applied to the quantum Hall effect (von Klitzing *et al* 1980).

The usual approach is that due to Pruisken *et al* (see Pruisken 1987) in which a $\sigma_{xx} - \sigma_{xy}$ renormalisation group flow diagram is derived from a non-linear σ model by including an extra, topological θ term, related to σ_{xy}. The Pruisken flow diagram is periodic in σ_{xy} and contains a single fixed point for each Landau level located at half integer σ_{xy} and at σ_{xx} of order unity (in units of e^2/h). This saddle point directs the flow towards σ_{xy} = integer and $\sigma_{xx} = 0$; except at half integer σ_{xy}, where the flow is towards the fixed point. This diagram seems to be confirmed by experimental results deriving a temperature driven flow diagram (Wei *et al* 1985, Kawaji and Wakabayashi 1987).

On the other hand, numerous numerical results, especially those by Ando and Aoki (Aoki 1987), but also by Chalker and Coddington (1988) and by the present author (Schweitzer *et al* 1984) are difficult to reconcile with the Pruisken theory. Perhaps surprisingly, although these numerical approaches have very different starting points, they agree quantitatively with one another. This suggests that the behaviour they predict is model independent. Aoki claims that the numerical results are inconsistent with the two-parameter scaling theory. It is not clear to me whether he is referring to two-parameter scaling theories *per se*, or to the Pruisken theory in particular. Indeed, Aoki and Ando (Aoki 1987) have calculated their own $\sigma_{xx} - \sigma_{xy}$ flow diagram, which contains no crossing flow lines, and is therefore consistent with the basic principle of two-parameter scaling.

More recently Clark *et al* (1987) have measured a temperature driven flow diagram for the fractional effect that seems to be consistent with a very speculative theoretical diagram published some years ago (Laughlin *et al* 1985). In this diagram the fractionally charged quasiparticles (Laughlin 1983, Halperin 1984, Haldane 1983) behave

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as the electrons do in the integer effect, but with the $\sigma_{xx} - \sigma_{xy}$ flow diagram rescaled to reflect the fractional charges. In order to derive a consistent diagram Laughlin *et al* (1985) were forced to include an extra unstable fixed point between two stable fixed points on each half integer (or equivalent) line. There was no other physical justification for the inclusion of this feature.

These very different approaches are represented in terms of flow diagrams. The length scale with respect to which the flow is measured may be the physical size of the system or the width of a strip, in numerical work. In the field theory, however, it appears as the lower limit of an integral over reciprocal space, whereas for comparison with experiment one must think in terms of the inelastic scattering length. In addition, different methods may involve different averaging procedures or even calculations of different but related quantities. Hence one should not expect to be able to plot the results of these different approaches on top of one another. However, the main topological features of the flow, such as fixed points and attractors, should not depend on these details. Quantities associated with these topological features, such as critical exponents, should also be universal.

In this paper I shall present some numerical results and shall attempt to analyse them in terms of a two-parameter scaling theory. I shall show that the results are consistent with a flow diagram containing two fixed points rather than one on each half integer line. Hence the extra fixed point in the generalised diagram can be attributed to the behaviour of non-interacting electrons in a magnetic field rather than to the effects of the interactions.

2. Numerical calculations

The calculations were carried out using a tight-binding model with the magnetic field represented by Peierls factors (Harper 1955), thus

$$\mathscr{H} = \sum_{mn} \epsilon_{mn} |mn\rangle \langle mn| + \sum_{mnm'n'} V_{mnm'n'} |mn\rangle \langle m'n'| \tag{1}$$

where

$$V_{mnm'n'} = \begin{cases} 1 & m = m' \quad \text{and } n = n' \pm 1 \\ \exp(\pm \mathrm{i} 2\pi n/L_{\mathrm{B}}) & m = m' \pm 1 \text{ and } n = n' \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The (m, n) represents points on a square lattice and ϵ is a random number chosen from a rectangular distribution of width W $(-\frac{1}{2}W < \epsilon \leq \frac{1}{2}W)$.

Using the transfer matrix method (MacKinnon and Kramer 1983) the smallest Lyapunov exponent, $\alpha = 1/\lambda_M$, was calculated for long strips with periodic boundary conditions and width M, where $4 \leq M \leq 64$. This exponent can be interpreted as the inverse localisation length of states on an infinitely long strip. Only a small sample of the results are presented here, with $L_{\mathrm{B}} = 8$ and $W = 0.5$. The energies are chosen to scan the lowest Landau level, which is located at about $E = -3.29$ in the absence of disorder.

Experience from the simpler localisation problem without a magnetic field (MacKinnon and Kramer 1983) has shown that the appropriate quantity to study is the renormalised localisation length $\Lambda = \lambda_M/M$, which displays one-parameter scaling behaviour. In other words, the changes in Λ with increasing M can be expressed as a function of Λ alone. The results are summarised in figure 1, where Λ is plotted against $\delta E\,|\delta E|\,M$ in the form of a flow diagram for increasing M. Here δE is the deviation of the energy from the apparent centre of the Landau level ($E_0 = -3.292$ when $W = 0.5$). The points for different M corresponding to a single energy are joined by a cubic spline curve and the individual points are represented by arrowheads in the direction of the local flow (ie the gradient required to calculate the spline). The arguments leading to this unconventional way of representing the data are discussed in the next section. Here it suffices to note that the flow lines rarely cross. Those crossings which are present are at small system sizes and can be accounted for by the statistical errors in the data (relative error 1%).

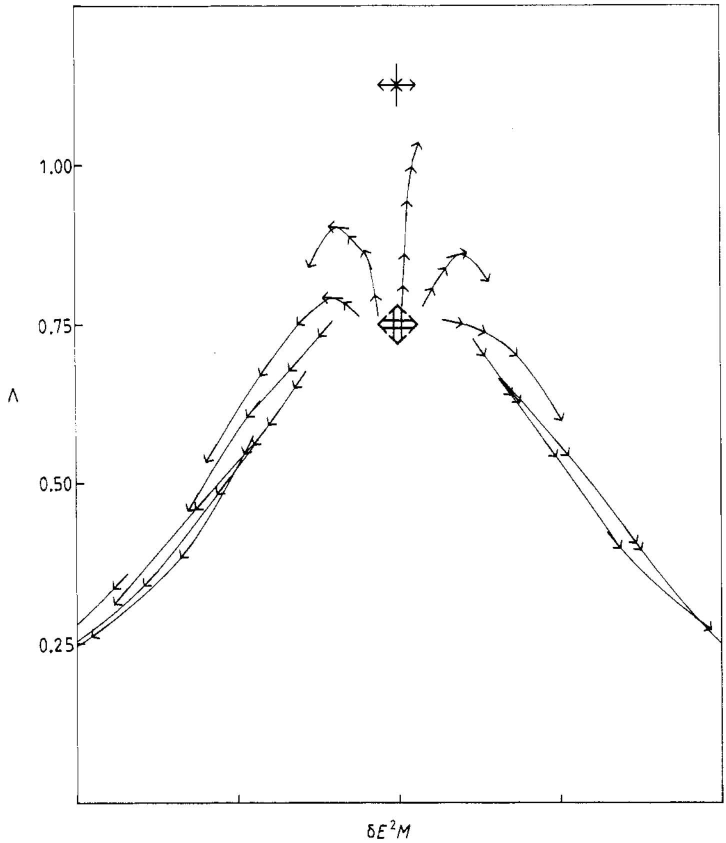

Figure 1. Renormalised localisation length Λ against $\delta E\,|\delta E|\,M$, where δE is the deviation of the energy from the centre of the lowest Landau level ($E_0 = -3.292$) and M is the width of the strip. The disorder is $W = 0.5$ and magnetic field given by $L_B = 8$. The energy changes in steps of 0.01 on either side of the central (almost vertical) line at $E = -3.29$. The data points are represented by arrows in the direction of increasing system size ($M = 4, 8, 16, 32, 64$) as calculated from a cubic spline fit to the data for a single energy (continuous lines). The crossed arrows are *guides to the eye* marking the two fixed points.

Figure 1 has all the characteristics of a renormalisation group flow diagram. Indeed, it looks rather similar to the $\sigma_{xx} - \sigma_{xy}$ diagram of Aoki and Ando (Aoki 1987) There is clearly an unstable fixed point at about $\Lambda \approx 0.75$ and possibly a saddle point at the top of the diagram. The diagram is consistent with two-parameter scaling but not with the Pruisken theory.

3. Data Analysis

Let $\delta\Lambda$ and $\delta\Xi$ be the deviations from the fixed point in the direction of Λ and in the direction of the unknown second parameter, Ξ, respectively. Since the data in figure 1 are symmetric around $\delta E = 0$, the general equation for the flow around the fixed point reduces to

$$\frac{\mathrm{d}\,\delta\Lambda}{\mathrm{d}\ln M} = \alpha\,\delta\Lambda + \gamma\,\delta\Lambda^2 + \epsilon\,\delta\Xi^2 \tag{3a}$$

$$\frac{\mathrm{d}\,\delta\Xi}{\mathrm{d}\ln M} = \beta\,\delta\Xi + \delta\,\delta\Lambda\,\delta\Xi. \tag{3b}$$

The unknown quantity $\delta\Xi$ can be eliminated to give a second-order differential equation which can then be linearised in $\delta\Lambda$ to give

$$\frac{\mathrm{d}^2\,\delta\Lambda}{\mathrm{d}\ln M^2} - (\alpha + 2\beta)\frac{\mathrm{d}\,\delta\Lambda}{\mathrm{d}\ln M} + 2\alpha\beta\,\delta\Lambda = 0 \tag{4}$$

which has the general solution

$$\delta\Lambda = AM^{\alpha} + BM^{2\beta}. \tag{5}$$

Since Λ must be an analytical and symmetric function of δE for all finite M, its behaviour near the fixed point can be described by

$$\Lambda = \Lambda^* + (a + b\delta E^2)M^{\alpha} + (c + d\delta E^2)M^{2\beta}. \tag{6}$$

This general form can be fitted to the raw numerical data to find the seven unknowns. There is more than enough data for this to be a valid procedure. We can distinguish between the otherwise indistinguishable terms in (6) by considering the behaviour at $\delta E = 0$. Here the α term must dominate, since $\delta\Xi = 0$ in (3). Thus we expect to find $c = 0$.

In practice $2\beta = 1.0 \pm 0.05$, $c \approx 0.01d$ when the units of δE are chosen such that δE is typically of order unity. Since the β term in (6) can also be written in terms of M/ξ, where ξ is the localisation length, $2\beta = 1$ implies $\xi \sim 1/\delta E^2$ in agreement with other results (Aoki 1987, Chalker and Coddington 1988). The other parameters have rather large error bars, and the fitted values are rather sensitive to the details of the fitting procedure, in particular α, which seems to be very small. This may reflect the presence of another fixed point just above the range of the present data or the presence of a so–called irrelevant variable whose effect has not yet died out at the system sizes studied.

The above analysis provides the justification for plotting Λ against $\delta E\,|\delta E|\,M$ in figure 1 in order to obtain a two-dimensional flow diagram to describe the behaviour of Λ.

4. Conclusions

So far the discussion has been in terms of the renormalised localisation length Λ and the unknown second scaling variable Ξ. There is no simple mapping between these and $\sigma_{xx} - \sigma_{xy}$ and it is unclear whether such a mapping should even exist in principle, as it is impossible to represent σ_{xy} in terms of states at the Fermi energy alone in the presence of periodic boundary conditions. Nevertheless, the behaviour of σ_{xx} can be represented in terms of Fermi energy quantities and it may be assumed that the flow diagram for $\sigma_{xx} - \sigma_{xy}$ will be broadly similar to that for $\Lambda - \Xi$. In particular, the unstable fixed point and probably a saddle point should be important features of both. However, it may not be valid to draw any inference about the behaviour of σ_{xy} from that of Ξ.

The Pruisken theory involves an expansion in $1/\sigma_{xx}$ that may break down at small values of σ_{xx}. The presence of an unstable fixed point at smaller σ_{xx} than Pruisken's saddle point is thus not necessarily inconsistent with his results. In fact, as mentioned in the introduction, such a fixed point is a necessary part of the generalisation of Pruisken's flow diagram to the fractional quantum Hall effect. The results presented here together with those of Aoki and Ando (Aoki 1987) and of Chalker and Coddington (1988) are consistent with such a picture.

A full understanding of the quantum Hall effect requires an understanding of the role of both disorder and interactions. I have shown that the behaviour of non-interacting electrons in a magnetic field is described by a pair of fixed points, in the renormalisation group description. If this unit is repeated on a different scale to describe also the behaviour of the fractionally charged quasiparticles in the presence of disorder, the same diagram results as in the prediction of Laughlin *et al* (1985). The main difference is in the interpretation. In the picture of Laughlin *et al* the unstable fixed point is related to the many-body effects that lead to the fractional states. In the present picture the unstable fixed point is a single particle effect, related to the integer quantum Hall effect.

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